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Two-dimensional Ising model with multispin interactions

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Abstract. The two-dimensional Ising model in the rectangular lattice is generalised to include m-spin interactions in one direction and two-spin interactions in the other. This model is self-dual and the critical line is the same as in the conventional Ising model with $m=2$.

The $m=3$ model is solved in a generalised mean-field approximation where the transition is first order. Using a phenomenological renormalisation group approach, approximate values of the thermal and magnetic exponents y_t and y_h are obtained for $m=3$ and $m=4$. The $m=3$ results together with symmetry arguments suggest that this model belongs to the same class of universality as the Baxter–Wu and four-state ferromagnetic Potts models. When $m=4$ the transition is probably first order.

1. Introduction

Multispin interactions may lead in lattice statistical models to a rich variety of critical behaviour. Let us mention the exactly solved eight-vertex model with continuously varying exponents (Baxter 1972) which may be formulated as an Ising model with two- and four-body interactions (Wu 1971, Kadanoff and Wegner 1971), the Baxter–Wu model (Baxter and Wu 1974, Baxter 1974), an Ising model with three-spin interactions on every face of the triangular lattice which belongs to the same class of universality as the four-state Potts model (Potts 1952) according to the den Nijs conjecture (den Nijs 1979). In other cases universal behaviour is observed as in the ferromagnetic q-state Potts model with multispin interactions which was introduced to describe site percolation in the $q=1$ limit (Giri *et al* 1977, Kunz and Wu 1978). Multisite terms were also used in more physical models to describe the phase diagrams of metallic alloys (Sanchez and de Fontaine 1981) or exotic structures in Heisenberg magnets (see for example Nagaev (1982) for a review).

In the present work we introduce a new class of anisotropic two-dimensional (2D) Ising models with m-spin interactions in the spatial (horizontal) direction and the usual two-spin Ising interaction in the temporal (vertical) direction on the rectangular lattice‡ (figure 1). The interest of such a formulation lies in the fact that, as in the q-state Potts model, one may study the evolution of the critical behaviour as a function of a parameter which here is the number of spins m entering the multisite interaction. The model may be further generalised by introducing n-spin interactions in the

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‡ A preliminary account of this work was given in Turban (1982a). The 1D quantum version was presented in Turban (1982b) and in Penson *et al* (1982).

temporal direction and/or Potts instead of Ising spins (Turban 1982c and Turban and Debierre 1982).

These models are self-dual and when the transition is unique, the free energy is singular on the critical line

$$\sinh(2K_x)\sinh(2K_\tau) = 1 \tag{1.1}$$

which is m-independent. In equation (1.1) K_x and K_τ are the interactions (in units of $k_B T$) in the spatial and temporal directions. The outline of the paper is as follows. In § 2 we give a mean-field treatment of the $m = 3$ model. The self-duality and exact results for the critical point are established in § 3. The $m = 2$, 3 and 4 models are studied using the phenomenological renormalisation group method (Nightingale 1976) and finite-size scaling in § 4. We end with a discussion of these results in § 5.

2. Mean-field approximation

Mean-field theory is often a useful first step in the study of a phase transition. Although it may be qualitatively wrong in predicting the order of the phase transition in low-dimensional systems, it gives exact information about the nature of the transition above the critical dimensionality d_c (which here will depend on m as it depends on q in the Potts model) when fluctuations become negligible.

Let the Hamiltonian of the system be written

$$-\beta H_m = K_\tau \sum_l s_i s_j + K_x \sum_L \prod_{k=1}^{m} s_k + H \sum_i s_i \tag{2.1}$$

where the $\{s\}$ are Ising spins ($s = \pm 1$) associated with the N sites of a rectangular lattice; the first sum runs over the N simple links l in the temporal direction and the second on the N multiple links L between m successive spins in the spatial direction; $h = H/\beta$ is the external field. A natural generalisation of the usual mean-field theory consists in working with a cluster of $m-1$ spins $(s_1, s_2, \ldots s_{m-1})$ in the spatial direction (figure 1). This will leave us with $m-1$ selfconsistent equations for the variational parameters so that m has to be specified. We give here a solution of the first non-trivial case where $m = 3$.

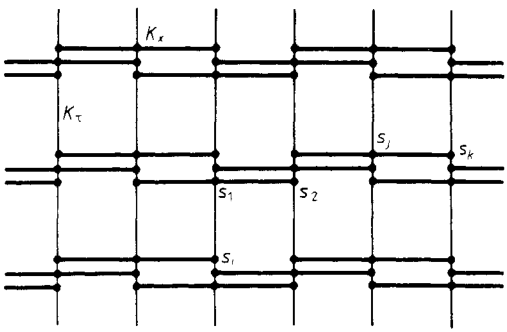


Figure 1. Ising model with m-spin interactions K_x and two-spin interactions K_τ on the rectangular lattice ($m = 3$). In the mean-field approximation one works on a two-spin cluster (s_1, s_2) in the spatial direction.

Let us introduce two variational parameters μ and ϕ (see for example Sherrington (1980)) and make the following substitutions (figure 1).

$$K_\tau s_1 s_i \to \mu K_\tau (s_1 - \mu/2) \tag{2.2a}$$

$$K_x s_1 s_2 s_j \to \mu K_x (s_1 s_2 - \phi/2) \tag{2.2b}$$

$$K_x s_2 s_j s_k \to \phi K_x (s_2 - \mu/2) \tag{2.2c}$$

where subtractive terms are included to avoid double counting of the interactions in the cluster free energy. The cluster effective Hamiltonian

$$-\beta H_{12}(\mu, \phi) = K_{\text{eff}} s_1 s_2 + H_{\text{eff}}(s_1 + s_2) \tag{2.3}$$

involves an effective interaction

$$K_{\text{eff}} = 2\mu K_x \tag{2.4}$$

and an effective field

$$H_{\text{eff}} = 2\mu K_\tau + \phi K_x + H. \tag{2.5}$$

Minimisation of the cluster free energy

$$\beta f(\phi, \mu) = 2(\phi\mu K_x + \mu^2 K_\tau) - \ln\{\text{Tr}\exp[-\beta H_{12}(\phi, \mu)]\} \tag{2.6}$$

with respect to ϕ and μ leads to

$$\mu = \langle s_1 \rangle = \text{Tr}\, s_1 \exp(-\beta H_{12})/Z_{12} \qquad \phi = \langle s_1 s_2 \rangle = \text{Tr}\, s_1 s_2 \exp(-\beta H_{12})/Z_{12} \tag{2.7a, b}$$

with

$$Z_{12} = \text{Tr}\exp(-\beta H_{12}) = 4(\cosh^2 H_{\text{eff}} \cosh K_{\text{eff}} + \sinh^2 H_{\text{eff}} \sinh K_{\text{eff}}). \tag{2.8}$$

We get two selfconsistent equations

$$\mu = \frac{\tanh H_{\text{eff}}(1 + \tanh K_{\text{eff}})}{1 + \tanh^2 H_{\text{eff}} \tanh K_{\text{eff}}} \qquad \phi = \frac{\tanh K_{\text{eff}} + \tanh^2 H_{\text{eff}}}{1 + \tanh^2 H_{\text{eff}} \tanh K_{\text{eff}}} \tag{2.9a, b}$$

where K_{eff} and H_{eff} are given by equations (2.4) and (2.5). When $H = 0$ the trivial solution $\phi = \mu = 0$ is stable only for high enough temperatures. The non-trivial solution with a spontaneous magnetisation $\mu \neq 0$ was studied numerically on a model with isotropic couplings $K_x = K_\tau = K$. The two solutions exchange their stability at the critical point K_c for which

$$f(\mu, \phi) = f(0, 0). \tag{2.10}$$

The spontaneous magnetisation given in figure 2 is discontinuous at $K_c = 0.3065$ whereas the exact value is $K_c = 1/2 \ln(1 + \sqrt{2}) = 0.44069$ by duality. The transition is first order with $m = 3$ whereas it is second order with $m = 2$, the usual Ising model. We are in a situation similar to that of the ferromagnetic q-state Potts model, m and q playing similar roles. For any $m > 2$ one may expect the transition to change from second to first order at a critical dimensionality $d_c(m)$. This point will be further explored in the renormalisation group study (§ 4).

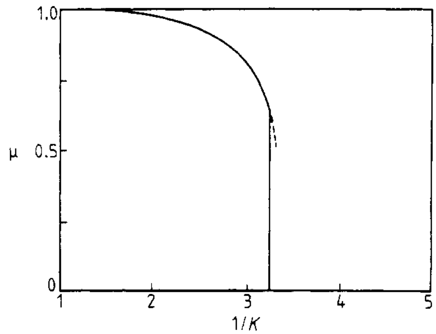


Figure 2. Spontaneous magnetisation of the $m = 3$ Ising model in the mean-field approximation.

3. Self-duality and related exact results

In this section we show that the self-duality property of the usual 2D Ising model with $m = 2$ (Wannier and Kramers 1941) is conserved for any m.

In the Hamiltonian of the system (2.1) with periodic boundary conditions, K_x and K_τ may be chosen positive without loss of generality in zero external field since the partition function Z_m is invariant under a change of sign of the interactions. A minus sign may be eliminated through a spin reversal on an appropriate sublattice.

The partition function

$$Z_m = \sum_{\{s\}} \exp(-\beta H_m) = \sum_{\{s\}} \prod_l \exp\left(K_\tau s_i s_j\right) \prod_L \exp\left(K_x \prod_{k=1}^{m} s_k\right) \tag{3.1}$$

may be written as

$$Z_m = \sum_{\{s\}} \prod_l \sum_{\lambda_\tau=0,1} C_{\lambda_\tau}(K_\tau)(s_i s_j)^{\lambda_\tau} \prod_L \sum_{\lambda_x=0,1} C_{\lambda_x}(K_x)\left(\prod_{k=1}^{m} s_k\right)^{\lambda_x} \tag{3.2}$$

where we made use of the identity (Savit 1980)

$$\exp\left(K \prod_k s_k\right) = \sum_{\lambda=0,1} C_\lambda(K)\left(\prod_k s_k\right)^{\lambda} \tag{3.3}$$

with

$$C_0(K) = \cosh K \qquad C_1(K) = \sinh K. \tag{3.4}$$

The spin products in (3.2) may be rearranged in order to group together all the s_i associated with the same site i. This leads to

$$Z_m = \sum_{\{\lambda\}} \prod_l C_{\lambda_\tau}(K_\tau) \prod_L C_{\lambda_x}(K_x) \sum_{\{s\}} \prod_i (s_i)^{\Sigma_i \lambda} \tag{3.5}$$

where the first sum is over the link variables $\lambda_x, \lambda_\tau = 0, 1$ and the exponent $\Sigma_i \lambda$ is a sum over the λ variables involving site i (figure 3).

Taking the trace over the spins s_i, we get

$$Z_m = 2^N \sum_{\{\lambda\}} \prod_l C_{\lambda_\tau}(K_\tau) \prod_L C_{\lambda_x}(K_x) \prod_i \delta_2(\Sigma_i \lambda) \tag{3.6}$$

where $\delta_2(n)$ is the Kronecker delta function mod 2.

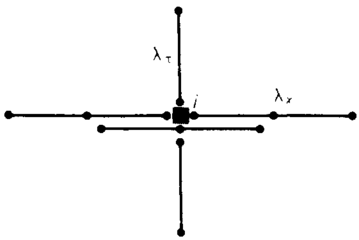


Figure 3. The exponent $\Sigma_i\lambda$ in equation (3.5) involves the two λ_τ and m λ_x associated with site i (here $m=3$).

The dual lattice is constructed by placing new Ising spins σ_i in the centre of the faces of the original lattice when m is even or in the middle of the temporal links l of the original lattice when m is odd (figure 4). These dual spins may be used to get a representation of the λ variables (figure 5)

$$\lambda_\tau(\sigma)=\tfrac{1}{2}\left(1-\prod_{k=1}^{m}\sigma_k\right) \qquad \lambda_x(\sigma)=\tfrac{1}{2}(1-\sigma_i\sigma_j) \tag{3.7a, b}$$

which automatically satisfies the Kronecker delta function in equation (3.6). In the duality operation the temporal and spatial directions are exchanged, a multiple (simple) link $L_{\rm d}(l_{\rm d})$ on the dual crosses a simple (multiple) link $l(L)$ on the original lattice. Since $\Sigma_i\lambda$ involves two λ_τ and m λ_x, it may be written as (figure 6)

$$\sum_i\lambda=(m+2)/2-\tfrac{1}{2}[(m+2)\text{ spin products}]. \tag{3.8}$$

Each dual spin enters two spin products in the bracket which may assume the following values

$$m+2-4\nu \qquad \nu\in N \qquad 0\leqslant\nu\leqslant \mathrm{I}(m/2)+1 \tag{3.9}$$

where $\mathrm{I}(x)$ means the integer part of x and the state with all the spins up corresponds to $\nu=0$.

Equations (3.8) and (3.9) give

$$\sum_i\lambda=2\nu \tag{3.10}$$

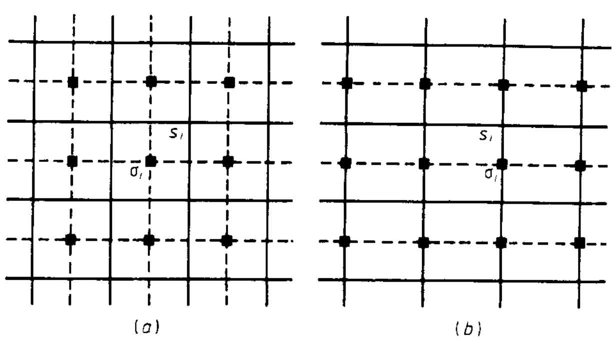


Figure 4. The original (full lines) and dual (broken lines) lattices when m is even (a) or odd (b).

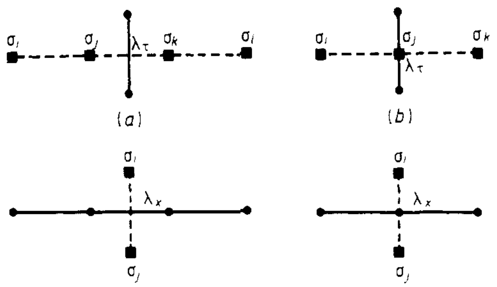


Figure 5. Correspondence between the λ_x and λ_τ variables on the original lattice (full lines) and the spin products on the dual lattice (broken lines) when m is even (a) or odd (b).

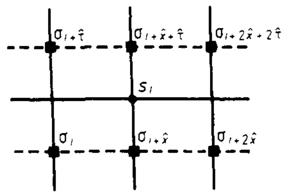


Figure 6. $\Sigma_i\lambda$ in the dual representation (equations (3.7a,b) in the text). In this case ($m = 3$) one gets $\Sigma_i\lambda = \frac{5}{2} - \frac{1}{2}[\sigma_i\sigma_{i+\hat\tau} + \sigma_{i+\hat x}\sigma_{i+\hat x+\hat\tau} + \sigma_{i+2\hat x}\sigma_{i+2\hat x+\hat\tau} + \sigma_i\sigma_{i+\hat x}\sigma_{i+2\hat x} + \sigma_{i+\hat\tau}\sigma_{i+\hat x+\hat\tau}\sigma_{i+2\hat x+\hat\tau}]$.

which is even as required. Any transformation of the dual spins leaving the energy of the dual configurations invariant does not change the $\{\lambda\}$ (figure 7). There are

$$g = \sum_{i=0}^{\mathrm{I}(m/2)} C_m^{2i} = 2^{m-1} \tag{3.11}$$

such transformations (figure 8) where g also gives the ground-state degeneracy of the model.

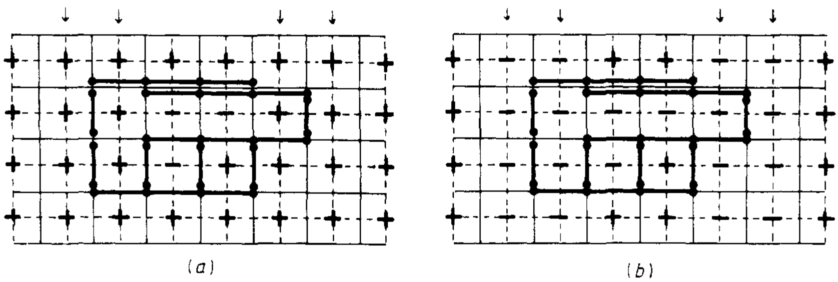


Figure 7. Two dual spin configurations leading to the same graph for the λ variables in the high-temperature expansion with $m = 4$. The dual configuration on (b) may be deduced from (a) by reversing the dual spins on the sublattice indicated by arrows. The Hamiltonian of the system is left invariant under this global symmetry transformation.

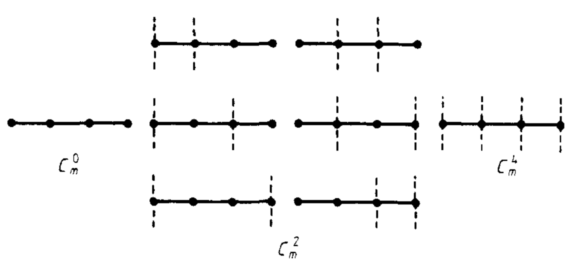


Figure 8. Symmetry transformations leaving the four-spin interaction invariant (reversed spins are indicated by broken lines). The degeneracy of the m-spin interaction is also the ground-state degeneracy of the system since a unique ground-state configuration may be built, spin by spin, from the four-spin configuration.

Collecting these results, the partition function becomes

$$Z_m = 2^{N+1-m} \sum_{\{\sigma\}} \prod_{L_d} C_{\lambda_\tau(\sigma)}(K_\tau) \prod_{l_d} C_{\lambda_x(\sigma)}(K_x) \tag{3.12}$$

or using the identity

$$C_\lambda(K) = \cosh K \exp(\lambda \ln \tanh K) \tag{3.13}$$

we get

$$Z_m(K_x, K_\tau) = 2^{1-m}(\sinh 2K_x \sinh 2K_\tau)^{N/2} Z_m(\tilde{K}_x, \tilde{K}_\tau) \tag{3.14}$$

where the dual couplings $\tilde{K}_x$ and $\tilde{K}_\tau$ are so defined that

$$\sinh 2\tilde{K}_\tau \sinh 2K_x = 1 \qquad \sinh 2\tilde{K}_x \sinh 2K_\tau = 1. \tag{3.15a, b}$$

It follows immediately that the model is self-dual and, when there is a unique phase transition, critical along the line

$$\sinh 2K_{x_c} \sinh 2K_{\tau_c} = 1 \tag{3.16}$$

which is left invariant by the duality transformation.

The most remarkable feature of this result is that although the nature of the transition may change with m, the number of spins involved in the interaction, the critical line itself is left unchanged.

Duality may be used to get some further information about the properties of the model at its critical temperature. Consider a model with isotropic values of the couplings

$$K_x = K_\tau = K = J/k_B T. \tag{3.17}$$

Equation (3.16) gives the critical temperature

$$\sinh^2(2J/k_B T_c) = 1 \tag{3.18}$$

and, in the thermodynamic limit, the free energy per site is

$$f(T) = \lim_{N\to\infty} -(k_B T/N) \ln Z_m(J/k_B T) = g(T) + (T/\tilde{T}) f(\tilde{T}) \tag{3.19}$$

where $\tilde{T}(T)$ is the dual temperature such that

$$\sinh(2J/k_B T) \sinh(2J/k_B \tilde{T}) = 1. \tag{3.20}$$

According to (3.14) and (3.19)

$$g(T) = -k_B T \ln \sinh(2J/k_B T) \tag{3.21}$$

so that $\tilde{T}(T_c) = T_c$ and $g(T_c) = 0$.

The entropy per site is

$$s(T) = -\frac{\partial g}{\partial T} + \frac{f(\tilde{T})}{\tilde{T}}\left(\frac{\partial \ln \tilde{T}}{\partial \ln T} - 1\right) + \frac{\partial \ln \tilde{T}}{\partial \ln T} s(\tilde{T}). \tag{3.22}$$

At the critical temperature

$$\left.\frac{\partial \ln \tilde{T}}{\partial \ln T}\right|_{T_{c+}} = \left.\frac{\partial \ln \tilde{T}}{\partial \ln T}\right|_{T_{c-}} = -1 \tag{3.23}$$

so that

$$s(T_{c+})+s(T_{c-})=-\frac{\partial g}{\partial T}\bigg|_{T_c}-2\frac{f(T_c)}{T_c} \tag{3.24}$$

and the internal energy per site is

$$\bar{u}=\frac{u(T_{c+})+u(T_{c-})}{2}=-\frac{T_c}{2}\frac{\partial g}{\partial T}\bigg|_{T_c}=-\sqrt{2}J. \tag{3.25}$$

When the transition is continuous, the internal energy keeps its Ising value at the critical temperature and this is also true for the average when the transition becomes first order, with higher m values.

4. Phenomenological renormalisation group approach

4.1. Phenomenological scaling (Nightingale 1976, Derrida and De Seze 1982)

Working on an infinite strip, n sites wide, the 2D system is critical only when we take the thermodynamic limit $n\to\infty$. It follows that any physical quantity $Q(k)$ with a singular part behaving near the critical point as

$$Q_\infty(K)\sim|K-K_c|^{x_q} \tag{4.1}$$

remains regular on the strip with a finite width. In the phenomenological scaling hypothesis, one assumes that a scaling function F_q exists, such that

$$Q_n(K)=Q_\infty(K)F_q[n/\xi_\infty(K)] \tag{4.2}$$

where $\xi_\infty(K)$ is the correlation length of the infinite system

$$\xi_\infty(K)\sim|K-K_c|^{-\nu}. \tag{4.3}$$

Compensation of the singularities of Q_∞ and ξ_∞ requires that

$$Q_n(K_c)\sim n^{-x_q/\nu} \tag{4.4}$$

for large n.

Considering two strips with width n and n' and two couplings K and K' such that

$$\xi_\infty(K)/\xi_\infty(K')=n/n' \tag{4.5}$$

equations (4.2) and (4.5) lead to

$$Q_n(K)/Q_n(K')=Q_\infty(K)/Q_\infty(K'). \tag{4.6}$$

When applied to the correlation length itself (4.6) gives the recursion of phenomenological scaling

$$\xi_n(K)/n=\xi_{n'}(K')/n' \tag{4.7}$$

telling us how the coupling is renormalised in the change of scale n/n'.

Since the infinite system is scale invariant at the critical point K_c an approximate value of the critical coupling $K_c(n, n')$ is obtained as the fixed point of the recursion relation. The correlation length exponent $\nu(n, n')$ follows from an expansion near the fixed point

$$1+\frac{1}{\nu(n,n')}=\frac{\ln[(\partial\xi_n/\partial K)_{K_c(n,n')}/(\partial\xi_{n'}/\partial K')_{K_c(n,n')}]}{\ln(n/n')}. \tag{4.8}$$

Other critical exponents may be obtained using (4.4)

$$-x_q(n,n')/\nu(n,n')=\ln(Q_n/Q_{n'})_{K_c(n,n')}/\ln(n/n') \tag{4.9}$$

at the approximate critical coupling and with the approximate correlation length exponent.

4.2. *Finite size scaling* (*Blöte and Nightingale 1982*)

$1/n$ may be considered as a relevant scaling field since it destroys the singularity when non-zero. Let $q^{(0)}$ be the singular part of the physical quantity Q with anomalous dimension y_q; $q^{(0)}$ is an homogeneous function of $1/n$ and of the other scaling fields u_i so that in a change of scale b

$$b^{y_q}q^{(0)}(u_1,u_2,\ldots,1/n)=q^{(0)}(b^{y_1}u_1,b^{y_2}u_2,\ldots,b/n). \tag{4.10}$$

For the derivatives of $q^{(0)}$ with respect to the scaling fields we get

$$\begin{aligned} q_n^{(k_1,k_2,\ldots)}&=\partial^{(k_1+k_2+\cdots)}q^{(0)}(u_1,u_2,\ldots,1/n)/\partial^{k_1}u_1\partial^{k_2}u_2\ldots \\ &=b^{-y_q+k_1y_1+k_2y_2+\cdots}q^{(k_1,k_2\ldots)}(b^{y_1}u_1,b^{y_2}u_2,\ldots,b/n)\end{aligned} \tag{4.11}$$

so that with $b=n$ the following large-n behaviour at the critical point ($u_1=u_2=\ldots=0$) is expected

$$q_n^{(k_1,k_2\ldots)}\sim n^{-y_q+k_1y_1+k_2y_2+\cdots}. \tag{4.12}$$

This is equivalent to the phenomenological scaling result (4.4) with $k_1=k_2=\ldots=0$, $y_t=1/\nu$ and $x_q=y_q/y_t$ as required.

When applied to the singular part of the free energy f with $y_f=d$ and to the inverse correlation length κ with $y_\kappa=1$, equation (4.12) gives

$$C_n\sim n^{-d+2y_t}\qquad \chi_n\sim n^{-d+2y_h} \tag{4.13a, b}$$

$$\kappa_n^{(t)}\sim n^{-1+y_t}\qquad \kappa_n^{(h)}\sim n^{-1+y_h} \tag{4.13c, d}$$

for the finite-size scaling at large n of the specific heat C, the susceptibility χ and the temperature and field derivatives $\kappa^{(t)}$ and $\kappa^{(h)}$ of the inverse correlation length at the critical point of the infinite system.

This analysis supposes that we work with the singular parts and ignores corrections to scaling. Better fits are obtainable using an expansion

$$Q_n=C_0+\sum_{i=1}^{\infty}c_in^{t_i} \tag{4.14}$$

where C_0 takes into account the regular part and t_1 gives the leading singularity (Blöte and Nightingale 1982).

4.3. Numerical methods

On a strip with periodic boundary conditions, length L in the temporal direction and width n, the partition function may be obtained by the transfer matrix method

$$Z_n = \mathrm{Tr}\, T_n^L = \sum_{i=1}^{2^n} \lambda_i^L \tag{4.15}$$

where the λ_i are the eigenvalues $(\lambda_1 > \lambda_2 > \ldots)$ of the transfer matrix

$$T_n(i,j) = \exp[-\beta H_m(i,j)] \tag{4.16}$$

where $H(i,j)$ is that part of the Hamiltonian (2.1) involving the spins of two successive rows i and j in the temporal direction with weight one-half for border interactions in order to avoid double counting in the matrix product. The free energy per spin f_n is given by

$$-\beta f_n = \lim_{L\to\infty} (1/nL) \ln Z_n = (\ln \lambda_1)/n \tag{4.17}$$

and the correlation length is

$$\xi_n = \kappa_n^{-1} = 1/\ln(\lambda_1/\lambda_2). \tag{4.18}$$

We use isotropic couplings $K_x = K_\tau = K$ and periodic boundary conditions in the spatial direction because free boundary conditions would produce prohibitive surface effects with narrow strips due to the m-spin interaction. The size of the transfer matrix may be reduced by taking the rotational symmetry into account.

In order to preserve the ground-state degeneracy of the problem we must use strips with a width which is a multiple of m, the number of spins entering the interaction in the spatial direction.

$K_c(n, n')$ and $\nu(n, n')$ are calculated with an accuracy of 10^{-7} and 10^{-5}. χ_n and C_n are obtained through numerical differentiation of λ_1 with six significant digits. The precision on λ_1 was increased in the calculation of the derivatives.

As a test, table 1 gives the values of $K_c(n, n')$, $\nu(n, n')$ and $\gamma(n, n')$ obtained through phenomenological scaling ((4.7), (4.8) and (4.9)) for $m = 2$, the usual Ising model. The results are in complete agreement with the work of Nightingale (1976). Table 2 gives the same quantities and $\alpha(n, n')$, the specific heat exponent, when $m = 3$. Table 3 gives y_t and y_h obtained through two-point fits of C, χ, $\kappa^{(t)}$ and $\kappa^{(h)}$ (4.13) at the exact critical coupling K_c, which is known by duality (§ 3). Table 4 gives the same results with $m = 4$. $\kappa_n^{(h)}$ could not be used in this case because it vanishes for even m values. Figures 9 and 10 give the field dependence of the susceptibility and the temperature dependence of the specific heat with increasing strip width for $m = 4$.

5. Discussion

Duality shows that the critical line does not depend on m; this results from a competition between the range of the interactions and the ground-state degeneracy; both are increasing with m, the first favouring order and the second disorder.

The main interest of this model lies in the change in the nature of the transition from second to first order as m is increased, a feature already encountered in the ferromagnetic q-state Potts model as a function of q. In a recent work on a quantum

Table 1. Critical coupling K_c, correlation length exponent ν and susceptibility exponent γ obtained through phenomenological scaling for the conventional Ising model ($m = 2$).

(n, n')	$K_c(n, n')$	$\nu(n, n')$	$\gamma(n, n')$
(4, 3)	0.430 8837	0.944 18	1.6340
(5, 4)	0.435 9532	0.964 39	1.6757
(6, 5)	0.438 2583	0.977 24	1.7027
(7, 6)	0.439 3099	0.984 58	1.7180
Exact values	0.440 6868	1.0	1.75

Table 2. The same as in table 1 for the $m = 3$ Ising model. The four-state Potts model values are added for comparison.

(n, n')	$K_c(n, n')$	$\nu(n, n')$	$\gamma(n, n')$	$\alpha(n, n')$
(3, 6)	0.418 2968	0.722 59	1.1847	0.963 04
(6, 9)	0.433 6138	0.726 94	1.2033	0.673 96
Potts $q = 4$		0.666 67	1.1667	0.666 67

Table 3. Thermal and magnetic exponents y_t and y_h obtained through finite-size scaling on the temperature and field derivatives of the inverse correlation length ($\kappa^{(t)}$ and $\kappa^{(h)}$), on the specific heat C and susceptibility χ. The den Nijs conjectures for the four-state Potts model are added for comparison as well as the Blöte and Nightingale values when possible.

(n, n')	$y_t(\kappa^{(t)})$	$y_t(C)$	$y_h(\kappa^{(h)})$	$y_h(\chi)$
(3, 6)	1.2411 (1.2787)	1.6729 (1.4514)	1.9535 (1.9049)	1.9591 (1.9152)
(6, 9)	1.2832	1.4690 (1.3962)	1.9198	1.9230 (1.8835)
Potts $q = 4$	1.5	1.5	1.875	1.875

Table 4. The same as in table 3 for the $m = 4$ Ising model. The exact values for a 2D discontinuity fixed point are added for comparison.

(n, n')	$y_t(\kappa^{(t)})$	$y_t(C)$	$y_h(\kappa^{(h)})$	$y_h(\chi)$
(4, 8)	1.4623	1.8994	—	1.9587
Discontinuity fixed point	2	2	2	2

q-state Potts version of the present model in $(1+1)$ dimensions (Turban and Debierre 1982) we got an approximate expression, through a $1/q$ expansion of the latent heat (Kogut 1980), for the curve $q_c(m)$ on which the transition changes from first to second order (figure 11). This curve gives the 2D behaviour of the classical formulation. We found $q_c(2) = 6$ and $q_c(3) = 3$ in the first-order approximation. A simple rescaling of these results by a factor $\frac{2}{3}$ gives the exact result $q_c(2) = 4$ (Baxter 1973) and suggests $q_c(3) = 2$.

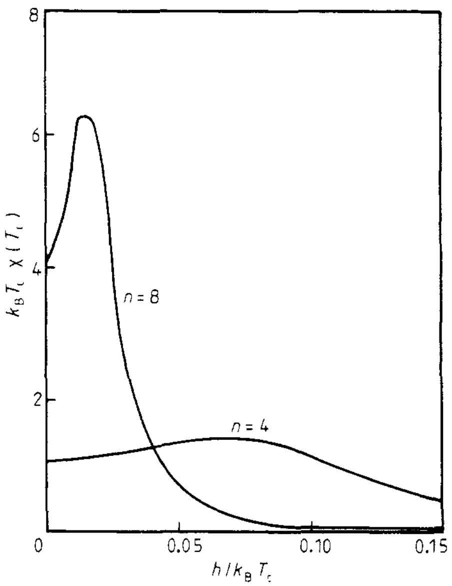


Figure 9. Field dependence of the susceptibility for $n=4$ and $n=8$ in the four-spin model. The curve is symmetric about $h=0$ for even values of m.

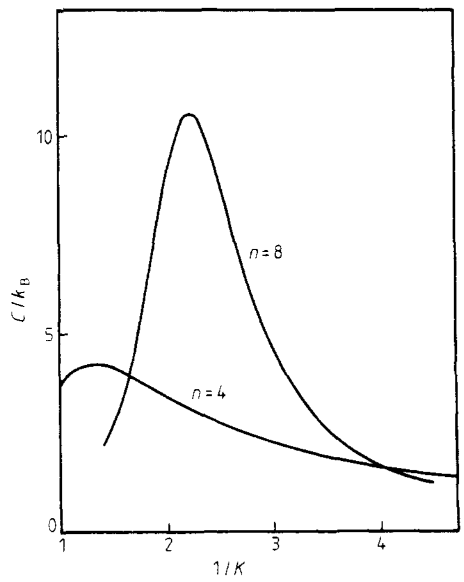


Figure 10. Temperature dependence of the specific heat for strips of width $n=4$ and $n=8$ and four-spin interactions.

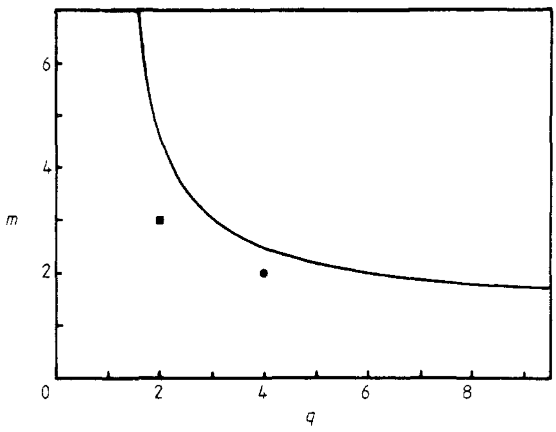
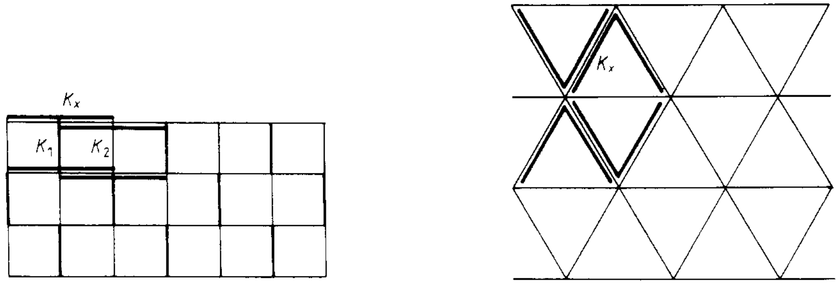


Figure 11. Frontier between first- and second-order transition in the (m, q) plane as given by a $1/q$ expansion of the latent heat of a Potts generalisation of the present model ($q=2$) in $(1+1)$ dimensions. The point ($m=2, q=4$) belongs to the exact frontier and the point ($m=3, q=2$) probably does also.

The four-state Potts model with $m=2$ and the Ising model with $m=3$ are discrete spin models with the same ground-state degeneracy $g=4$. Furthermore the Baxter–Wu model has also $g=4$ and is known to belong to the same class of universality as the four-state Potts model. As shown in figure 12, the Baxter–Wu and $m=3$ Ising models may be regarded as two limiting cases of the same model. All these facts led



(a) (b)

Figure 12. Generalised Ising model with three-spin interactions (*a*). In the limit $K_1 = K_2 = K_T$ we get the Ising model with three-spin interactions and the Baxter–Wu model corresponds to the limit $K_1 \to \infty$, $K_2 \to 0$ (*b*).

us to conjecture that the $m = 3$ Ising model belongs also to the class of universality of the $q = 4$ Potts models and gives the border between second- and first-order transitions in 2D.

One may verify in tables 2 and 3 that the renormalisation group results are consistent with this conjecture. In table 3 we compare our results with the $q = 4$ Potts values obtained by two-points fits of the Blöte and Nightingale results (1982) with the same strip width (3, 6) and (6, 9). We get the same behaviour with n although their (3, 6) results are closer to our (6, 9) results which may be due to the longer range of the interaction in our case leading to a slower convergence.

The $m = 4$ results in table 4 are comparable to the discontinuity fixed point values $y_t = 2$ and $y_h = 2$ in 2D.

Penson *et al* (1982) have independently studied the 1D quantum version of this model with the phenomenological scaling method. Working with larger sizes ($n = 3$, 6, 9, 12, 15 for $n = 3$ and $n = 4$, 8, 12 for $m = 4$) they were able to extrapolate their correlation length exponent values and got $\nu_\infty(3) \simeq 0.72$ which is quite close to the corresponding values of Blöte and Nightingale (1982) for the $q = 4$ Potts model and $\nu_\infty(4) \simeq 0.5$ in agreement with a discontinuity fixed point. The finite-size scaling relations given in equations (4.13*a*, *b*, *c*, *d*) are expected to break down for an infinite strip in the case of a discontinuity fixed point. According to the analysis of Blöte and Nightingale (1982) one should get then an exponential variation with n for large n instead of a power law. They have effectively found a linear variation of y_t and y_h with n when $q > 4$ for C_n, χ_n and $\partial^2\kappa_n/\partial t^2$ but also convergent estimates from $\kappa_t(q = 64)$ and κ_h ($q = 8, 64$) agreeing with the discontinuity fixed-point values like the $m = 4$ results of Penson *et al* (1982).

Finally let us mention that the m-spin Ising model may be changed, through an exact prefacing transformation (Berker 1975), into a more conventional model with 2^{m-1} states per site and nearest-neighbour interactions only by using cells of two spins in the spatial direction.

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